Int. J. Heat Mass Transfer. Vol. 5, pp. 439–444. Pergamon Press 1962. Printed in Great Britain.

A VARIATIONAL METHOD FOR COMBINED FREE AND FORCED CONVECTION IN CHANNELS

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(*Received* 9 *August* 1961, *and in revised form* 27 *November* 1961)

Abstract—A variational method is presented for studying the problems of combined free and forced convection in vertical channels. The method is applied to flow in a vertical rectangular channel and the results are compared with the available exact solutions. The method is further extended for circular ducts.

NOMENCLATURE

- A, area of cross-section;
- c_p, specific heat of the fluid at constant pressure;
- g, acceleration due to gravity;
- $\bar{h}$, average peripheral heat-transfer coefficient;
- k, fluid thermal conductivity;
- Nu, Nusselt number;
- Q, heat-generation rate;
- r, radius of the circular duct;
- R, dimensionless radius of the circular duct;
- Ra, Rayleigh number;
- t, temperature of the fluid;
- t_w, wall temperature;
- t_0, wall temperature at $z = 0$;
- T_m, dimensionless mean mixed temperature;
- w, component of fluid velocity in the z-direction;
- w_m, mean fluid velocity;
- ρ_0, constant density of the fluid;
- ρ_w, density of the fluid at the wall;
- μ, viscosity;
- β, coefficient of thermal expansion;
- θ_m, mean mixed temperature.

INTRODUCTION

THE effect of free convection on the forced heat transfer for fully developed laminar flow in vertical channels, has recently been realized as significant in many engineering problems of nuclear reactors, electrically heated vertical tubes and heat exchangers. These applications approximate a linearly varying wall temperature or uniform wall heat flux, rather than a uniform wall temperature. A number of theoretical investigations, by Ostrach [1], Hallman [2] and Han [3], are available in this direction. In all these cases, the solution of a fourth-order differential equation obtained by combining the momentum and energy equations, has been sought, which is, of course, very complicated. The recent work of Tao [4] suggests a method of solving such problems by introducing a complex function whose real and imaginary parts are related directly to the velocity and temperature fields, respectively. This combines the momentum and energy equations to give an inhomogeneous Helmholtz equation with homogeneous boundary conditions. Although the solution of this equation is less complicated, the computational work still seems to be lengthy.

In the present work, a variational principle has been developed for studying problems of this type. The inhomogeneous Helmholtz equation obtained by Tao [4] is replaced by an appropriate variational principle. In particular, the problem of flow in a vertical, rectangular channel is solved and the results are compared with those obtained by Han [3]. The principle is further extended for circular ducts. It is found that the variational approach greatly reduces the complexities of the solution of the problem.

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THE MOMENTUM AND ENERGY EQUATIONS

Consider a steady laminar flow of a liquid in a vertical channel under the joint influence of pressure-gradient and buoyancy forces. It is assumed that the flow velocity $w(x, y)$ is entirely parallel with the channel axis and is independent of distance z along the channel. As a consequence, the continuity equation, $\partial w/\partial z = 0$, is satisfied identically. The fluid properties, except for the density, are taken as constant in considering the buoyant effect. Viscous dissipation is neglected and the heat input from the boundary to the fluid is constant in the flow direction, i.e. the wall temperature is linear:

$$t_w = t_0 + Cz, \quad \frac{\partial t_w}{\partial z} = C.$$

The reason one can usefully seek an asymptotic solution of the problem, in which, $v = \{0, 0, w(x, y)\}$ and the distribution of buoyancy force is independent of z, is precisely because the wall temperature is taken as linear. Also, the rate of internal heat generation per unit volume, Q, is assumed to be uniform everywhere in the fluid. The flow is in the vertical upward direction, along the positive z-axis. Under these conditions, the equations of motion and temperature distribution are ([3], [4])

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{1}{\mu}\rho_0 g\beta\theta = \frac{1}{\mu}\left[\frac{\partial p}{\partial z} + g\rho_w\right] \quad (1)$$

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = \rho_0 c_p C w/k - Q/k \quad (2)$$

where θ is defined as a temperature-difference function

$$\theta = (t - t_w). \quad (3)$$

The boundary conditions are

$$\left.\begin{array}{l}\theta = 0\\ w = 0\end{array}\right\} \text{ at the wall.} \quad (4)$$

With the following dimensionless quantities

$$\left.\begin{array}{l}X = x/D,\\ Y = y/D,\\ T = k\theta/\rho_0 c_p C w_m D^2\\ W = w/w_m,\\ E = -\left(\dfrac{\partial p}{\partial z} + g\rho_w\right) D^2/\mu w_m,\\ F = Q/\rho_0 c_p C w_m\end{array}\right\} \quad (5)$$

where D is the equivalent or the hydraulic diameter, w_m is the mean fluid velocity and E is proportional to the net pressure-gradient force along the channel, the equations (1) and (2) are reduced to

$$\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} + Ra\, T = -E \quad (6)$$

$$\frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2} - W = -F \quad (7)$$

$$Ra \text{ (Rayleigh number)} = \rho_0^2 g c_p C\beta D^4/k\mu.$$

The complex functions

$$\Phi = W + i\epsilon^2 T, \quad G = F - i\epsilon^{-2}E \quad (8)$$

are introduced, where $\epsilon^4 = Ra$, and equations (5) and (6) are combined into

$$\frac{\partial^2 \Phi}{\partial X^2} + \frac{\partial^2 \Phi}{\partial Y^2} - i\epsilon^2\Phi = -i\epsilon^2 G. \quad (9)$$

The boundary conditions (4) become

$$\Phi = 0, \text{ on the boundary.} \quad (10)$$

Equation (9) is an inhomogeneous Helmholtz wave equation, with homogeneous boundary condition (10).

For axially symmetric flow in a vertical circular pipe, equation (9) becomes

$$\frac{\partial^2 \Phi}{\partial R^2} + \frac{1}{R}\frac{\partial \Phi}{\partial R} - i\epsilon^2\Phi = -i\epsilon^2 G, \; R = r/D \quad (11)$$

with $\Phi = 0$ on the boundary and R denoting the dimensionless radius of the pipe.

MEAN MIXED TEMPERATURE AND NUSSELT NUMBER

The mean mixed temperature may be defined as

$$\theta_m = \int_A \theta w \, dx\, dy / \int_A w \, dx\, dy. \quad (12)$$

In the dimensionless form it is

$$T_m = \int_A TW\,dX\,dY / \int_A W\,dX\,dY$$
$$= \frac{D^2}{2\epsilon^2 A} I_m \{\int_A \Phi^2\,dX\,dY\}. \quad (13)$$

The Nusselt number is defined in the usual manner as

$$Nu = \bar{h}D/k \quad (14)$$

where $\bar{h}$ is the average peripheral heat-transfer coefficient

$$\bar{h} = \left[\oint k \frac{\partial\theta}{\partial n}\,ds\right] \frac{1}{(\text{perimeter})\,\theta_m}. \quad (15)$$

The Nusselt number in the dimensionless form thus becomes

$$Nu = \frac{r_h}{D} \frac{1-F}{T_m} \quad (16)$$

where the hydraulic radius r_h is defined as the cross-sectional area divided by the circumference.

THE VARIATIONAL PROCEDURE

It is now necessary to formulate a variational principle which should be equivalent to equations (9) and (10). To this end, consider a variational integral

$$I = \iint_A f(X, Y, \Phi, \Phi_X, \Phi_Y)\,dX\,dY \quad (17)$$

where $\Phi(X, Y)$ is the unknown function. The necessary condition that the integral (17) be stationary is that its first variation vanishes, $\delta I = 0$. This requires that the integrand should satisfy the Eüler equation

$$\frac{\partial}{\partial X}\left(\frac{\partial f}{\partial \Phi_X}\right) + \frac{\partial}{\partial Y}\left(\frac{\partial f}{\partial \Phi_Y}\right) - \frac{\partial f}{\partial \Phi} = 0. \quad (18)$$

The function f is chosen in such a way that its insertion in (18) gives the required differential equation in Φ.

The Ritz method is then applied to determine Φ.

In order to derive (9) from (18), take

$$f = \left[\left(\frac{\partial\Phi}{\partial X}\right)^2 + \left(\frac{\partial\Phi}{\partial Y}\right)^2 - m\Phi^2 + 2mG\Phi\right],$$
$$m = -i\epsilon^2. \quad (19)$$

It is easy to verify that the choice of f is a proper one. The variational integral corresponding to the differential equation (9) is

$$I = \iint_{\text{area}}\left[\left(\frac{\partial\Phi}{\partial X}\right)^2 + \left(\frac{\partial\Phi}{\partial Y}\right)^2 - m\Phi^2 + 2mG\Phi\right] dX\,dY. \quad (20)$$

For a rectangular duct of sides $2a$, $2b$, expression (20) becomes

$$I = \int_{-1/2}^{+1/2}\int_{-\alpha}^{\alpha}\left[\left(\frac{\partial\Phi}{\partial X}\right)^2 + \left(\frac{\partial\Phi}{\partial Y}\right)^2 - m\Phi^2 + 2mG\Phi\right] dX\,dY \quad (21)$$

the equivalent diameter D being taken as $2a$ and $\alpha = (b/2a)$. The variational integral corresponding to equation (18) for the circular pipe of radius r, is

$$I = \int_0^{1/2}\left[\left(\frac{\partial\Phi}{\partial R}\right)^2 - m\Phi^2 + 2mG\Phi\right] R\,dR. \quad (22)$$

The correspondence between the variational integrals and the differential equations is subject to certain boundary conditions being satisfied. In the present problem, the physical requirement that $\Phi = 0$ on the boundary assures this correspondence.

APPLICATION TO THE RECTANGULAR DUCT

According to the Ritz method, when a function $\Phi(X, Y)$ is to be determined, the procedure consists in assuming that the desired extremal of the integral (21) can be approximated by a linear combination of n suitably chosen functions.

As a first approximation, we select a simple polynomial

$$\Phi = (X^2 - \tfrac{1}{4})(Y^2 - \alpha^2)[C_0 + C_1(X^2 + Y^2)] \quad (23)$$

C_0 and C_1 being the constants to be determined. The factors $(X^2 - \frac{1}{4})$ and $(Y^2 - \alpha^2)$ in each of the two terms of (23) satisfy the condition of Φ being zero at the boundaries $X = \pm\frac{1}{2}, Y = \pm\alpha$.

Substituting equation (23) into the variational integral (21) and integrating gives

$$I = C_0^2\left[\frac{16}{45}\alpha^5 - \frac{8}{225}m\alpha^5 + \frac{4}{45}\alpha^3\right] + C_0C_1\left[\frac{32}{315}\alpha^7 - \frac{16}{1575}m\alpha^7 + \frac{16}{225}\alpha^5 - \frac{4}{1575}m\alpha^5 + \frac{2}{315}\alpha^3\right]$$

$$+ C_1^2\left[\frac{16}{945}\alpha^9 - \frac{8}{4725}m\alpha^9 + \frac{52}{1575}\alpha^7 - \frac{4}{11025}m\alpha^7 + \frac{13}{1575}\alpha^5 - \frac{1}{9450}m\alpha^5 + \frac{1}{3780}\alpha^3\right]$$

$$+ \frac{4}{9}\alpha^3 mGC_0 + \frac{4}{45}mGC_1\left(\alpha^5 + \frac{\alpha^3}{4}\right). \tag{24}$$

To find the values of C_0 and C_1 which will make I a minimum, we differentiate the above expression with respect to C_0 and C_1, respectively, and set the resulting equations to zero. This gives

$$\frac{\partial I}{\partial C_0} = 0 \rightarrow C_0\left[\alpha^5\left(\frac{8}{5} - \frac{4}{25}m\right) + \frac{2}{5}\alpha^3\right] + C_1\left[\alpha^7\left(\frac{8}{35} - \frac{4}{175}m\right) + \alpha^5\left(\frac{4}{25} - \frac{m}{175}\right) + \frac{\alpha^3}{70}\right] + \alpha^3 mG = 0 \tag{25}$$

$$\frac{\partial I}{\partial C_1} = 0 \rightarrow C_0\left[\frac{16}{315}\alpha^7\left(2 - \frac{m}{5}\right) + \frac{4}{225}\alpha^5\left(4 - \frac{m}{7}\right) + \frac{2}{315}\alpha^3\right]$$

$$+ C_1\left[\frac{16}{945}\alpha^9\left(2 - \frac{m}{5}\right) + \frac{8}{1575}\alpha^7\left(13 - \frac{m}{7}\right) + \frac{1}{1575}\alpha^5\left(26 - \frac{m}{3}\right) + \frac{\alpha^3}{1890}\right]$$

$$+ \frac{4}{45}mG\left(\alpha^5 + \frac{\alpha^3}{4}\right) = 0. \tag{26}$$

For a square duct of side $2a$, we have $\alpha = \frac{1}{2}$. The equations for C_0 and C_1 derived from (25) and (26) are

$$C_0[0{\cdot}1 - 0{\cdot}005m] + C_1[0{\cdot}00857143 - 0{\cdot}000357143m] + (0{\cdot}125)mG = 0 \tag{27}$$

$$C_0[0{\cdot}1714285 - 0{\cdot}007142857m] + C_1[0{\cdot}05238095 - 0{\cdot}00085034m] + 0{\cdot}25mG = 0. \tag{28}$$

Solution of this group of linear algebraic equations provides the following values of C_0 and C_1.

$$C_0 = G\left[\frac{0{\cdot}0044047616m + 0{\cdot}000017Ra}{-0{\cdot}0037687075 + 0{\cdot}000224489m + 0{\cdot}0000017007Ra}\right] \tag{29}$$

$$C_1 = G\left[\frac{0{\cdot}00357143m + 0{\cdot}000357143Ra}{-0{\cdot}0037687075 + 0{\cdot}000224489m + 0{\cdot}0000017007Ra}\right]. \tag{30}$$

Making the denominators of the constants C_0 and C_1 real gives

$$C_0 = G\left[\frac{0{\cdot}0289119Ra^2 + 924{\cdot}756Ra + 3{\cdot}67485Ra\,m - 16600m}{0{\cdot}00289238Ra^2 + 37{\cdot}58282Ra + 14203{\cdot}156}\right] \tag{31}$$

$$C_1 = G\left[\frac{0{\cdot}638Ra^2 - 571{\cdot}646Ra - 77{\cdot}8357Ra\,m - 14138m}{0{\cdot}00289238Ra^2 + 37{\cdot}58282Ra + 14203{\cdot}156}\right] \tag{32}$$

With these values of constants, the expression for Φ is completely known, and may be split into real and imaginary parts. The real part of Φ gives the velocity distribution and the imaginary part, divided by $Ra^{1/2}$, denotes the distribution of temperature. If $F = 0$, i.e. when there is no heat source present, we have from (8)

$$G = -i\epsilon^{-2}E = -iE/Ra^{1/2}. \tag{33}$$

This gives

$$\Phi = iE(X^2 - \tfrac{1}{4})(Y^2 - \tfrac{1}{4})\left[\frac{-Ra^{3/2}\{0\cdot0289119 + (X^2 + Y^2)(0\cdot638)\} - Ra^{1/2}\{924\cdot756 - (X^2 + Y^2)(571\cdot646)\}}{0\cdot00289238Ra^2 + 37\cdot58282Ra + 14203\cdot156}\right]$$
$$+ E(X^2 - \tfrac{1}{4})(Y^2 - \tfrac{1}{4})\left[\frac{-Ra\{3\cdot67485 - (X^2 + Y^2)(77\cdot8357)\} + \{16600 + (X^2 + Y^2)(14138)\}}{0\cdot00289238Ra^2 + 37\cdot58282Ra + 14203\cdot156}\right]. \tag{34}$$

For $Ra = \pi^4$ and $E = 35\cdot13$ [3], the equation (34) yields

$$\Phi = (X^2 - \tfrac{1}{4})(Y^2 - \tfrac{1}{4})[\{(X^2 + Y^2)(42\cdot656333) + 31\cdot879993\} + i\{(X^2 + Y^2)(9\cdot876153) - 17\cdot9835\}], \tag{35}$$

From (13), the dimensionless mean mixed temperature is

$$T_m = (1/2Ra^{1/2})I_m\{\textstyle\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \Phi^2 \, dX \, dY\}. \tag{36}$$

Using (35), we get

$$T_m = -\frac{1\cdot33786169}{2Ra^{1/2}} = -0\cdot067724. \tag{37}$$

The Nusselt number derived from (16) is

$$Nu = \frac{1}{4T_m} = 3\cdot6914. \tag{38}$$

Similarly, for $Ra = 10\pi^4$, $E = 90\cdot73$ ([3]), and the expression for Φ is

$$\Phi = (X^2 - \tfrac{1}{4})(Y^2 - \tfrac{1}{4})[\{(X^2 + Y^2)(152\cdot40888) + 22\cdot021748\} - i\{(X^2 + Y^2)(2\cdot683716) + 50\cdot36403\}]. \tag{39}$$

The dimensionless mean mixed temperature is

$$T_m = -0\cdot05923. \tag{40}$$

The Nusselt number is

$$Nu = 4\cdot2207. \tag{41}$$

A comparison of the results with the available exact solutions is given in Table 1. The agreement is good.

Table 1

Ra	*E*	Nusselt number: Present method	Nusselt number: Exact method [3]	Error (%)
π^4	35·13	3·6914	3·69	+0·04
$10\pi^4$	90·73	4·2207	4·27	−1·0

CONCLUSION

The present method gives the solutions in terms of simple polynomials. The heavy computational work as required in series solution is thus avoided without loss of accuracy in the results.

The possibility of having a zero Rayleigh number arises if either $\beta = 0$ or $C = 0$. From (2) it is clear that, if $C = 0$, there is zero heating in the axial direction. The energy equation thus becomes independent of the velocity field, and the flow reduces to normal Poiseuille pipe flow. When $\beta = 0$, the buoyant term in the momentum equation disappears and the problem reduces to that of forced convection alone. In both these cases, the variational principles can be formulated *ab initio* [5].

ACKNOWLEDGEMENTS

The author thanks Dr. S. D. Nigam for helpful suggestions. He is indebted to Professor M. N. Rao, Head of the Chemical Engineering Department, for providing a research scholarship and encouragement.

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Résumé—On présente ici une méthode "de variation" pour l'étude des problèmes de convection mixte, libre et forcée, dans des conduites verticales. Cette méthode est appliquée à un conduit rectangulaire vertical et les résultats comparés aux solutions exactes valables. La méthode est ensuite étendue aux conduites circulaires.

Zusammenfassung—Mit Hilfe einer Variationsmethode kann das Problem der kombinierten freien und erzwungenen Konvektion in senkrechten Rohren studiert werden. Die Methode ist auf die Strömung im senkrechten rechteckigen Kanal angewandt; die Ergebnisse werden mit verfügbaren exakten Lösungen verglichen. Die Methode ist auch auf kreisförmige Rohre erweitert.

Аннотация—Предлагается вариационный метод решения задач свободной и вынужденной конвекций при их совместном действии в вертикальных каналах. Этот метод применён к течению в вертикальном прямоугольном канале и результаты сравнены с уже имеющимися точными решениями. В дальнейшем указанный метод обобщается на круглые трубы.